CHARACTERISTICS OF COSMIC TIME

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The nature of cosmic time is illuminated using Hamilton-Jacobi theory for general relativity. For problems of interest to cosmology, one may solve for the phase of the wavefunctional by using a line integral in superspace. Each contour of integration corresponds to a particular choice of time hypersurface, and each yields the same answer. In this way, one can construct a covariant formalism where all time hypersurfaces are treated on an equal footing. Using the method of characteristics, explicit solutions for an inflationary epoch with several scalar fields are given. The theoretical predictions of double inflation are compared with recent galaxy data and large angle microwave background anisotropies.

I. INTRODUCTION

One obtains a better appreciation for the nature of cosmic time by studying the role of inhomogeneities in general relativity [1]- [6]. After all, a time-hypersurface represents the arbitrary manner in which one slices a 4-geometry. However, there are numerous problems associated with quantum aspects of time [7]. One should perhaps be content to consider the semiclassical limit of the quantum theory where one retains only the leading order terms in Planck's constant $\hbar$ that appear in the Wheeler-DeWitt equation. This approximation leads directly to the Hamilton-Jacobi (HJ) equation for general relativity. In the present work, superspace solution techniques of the HJ equation will be further advanced to include the case of several scalar fields interacting in a cosmological setting. These methods simplify the comparison of theoretical models with astronomical observations.

The problem of choosing a gauge in general relativity has complicated the interpretation as well as the application of the theory. For example, when solving Einstein's equations, one must make arbitrary choices for the space-time coordinates. Many of these decisions may be postponed or even avoided by solving the HJ equation for general relativity. Hamilton-Jacobi theory leads to a covariant description of the gravitational field [1]- [6]. It yields a generalization of earlier work on covariant cosmological perturbations by Ellis *et al* [8].

Already, HJ techniques have been applied to numerous problems in cosmology including: (1) a detailed computation of microwave background fluctuations and galaxy-galaxy correlations [1], [9] generated in the power-law inflation model [11] which arises either in induced gravity [10] or extended inflation [12]; (2) a relativistic approach to the Zel'dovich approximation [13] describing the formation of sheet-like structures during the matter-dominated era [14], [15], [16]; (3) attempts to recover the inflaton potential from cosmological observations [10], [17]; (4) a construction of inflation models that yield non-Gaussian primordial fluctuations [5] —- such models may alleviate the problems of large scale structure in the Universe [18].

In Sec. II, I set forth the HJ equation and the momentum constraint equation describing the interaction of two scalar fields with the gravitational field. The extension to more fields is straightforward. The object of chief importance is the generating functional $\mathcal{S}$ which is the phase of the wavefunctional in the semiclassical approximation. In Sec. III, a spatial gradient expansion is used to solve the HJ equation. The zeroth order term for the generating functional, $\mathcal{S}^{(0)}$, contains no spatial gradients. It describes the evolution of long-wavelength fields. The second term, $\mathcal{S}^{(2)}$, contains two spatial derivatives. Its evolution equation is a linear partial differential equation which may be simplified using the method of characteristics. In particular, by invoking a local change of variables, one may solve this equation by using a line integral in superspace. Different contours of integration correspond to different choices of the time-hypersurface, but they all lead to the same answer for $\mathcal{S}^{(2)}$. In Sec. IV, it is shown how to effectively sum an infinite subset of terms in the spatial gradient expansion. This generalizes the quadratic curvature approximation that was employed by Salopek and Stewart [1]. The resulting Riccati equations may be reduced to ordinary linear differential equations using matrix methods. In Sec. V, initial conditions are given for a period of cosmological inflation. In Sec. VI, some examples and solutions are given for inflation with two scalar fields. Using analytic methods, one is able to compute fluctuation spectra that give rise to galaxy clustering. Previously, one required a tedious numerical computation [10]. Theoretical models are constrained by large angle microwave background anisotropies and galaxy clustering data.

(Units are chosen so that $c = 8\pi G = 8\pi/m_P^2 = \hbar = 1$. The sign conventions of Misner, Thorne and Wheeler [19] will be adopted throughout.)

II. THE HAMILTON-JACOBI EQUATION FOR GENERAL RELATIVITY

For two scalar fields ϕ_a, $a = 1, 2$, interacting through Einstein gravity, the Hamilton-Jacobi equation,

$$0 = \mathcal{H}(x) = \gamma^{-1/2} \left[2\gamma_{ik}(x)\gamma_{jl}(x) - \gamma_{ij}(x)\gamma_{kl}(x) \right] \frac{\delta \mathcal{S}}{\delta \gamma_{ij}(x)} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}(x)} +$$

$$\sum_{a=1}^{2} \frac{1}{2} \gamma^{-1/2} \left(\frac{\delta \mathcal{S}}{\delta \phi_a(x)} \right)^2 + \gamma^{1/2} V[\phi_a(x)] + \left[-\frac{1}{2}\gamma^{1/2}R + \sum_{a=1}^{2} \frac{1}{2}\gamma^{1/2}\gamma^{ij}\phi_{a,i}\phi_{a,j} \right] , \tag{2.1a}$$

and the momentum constraint equation,

$$0 = \mathcal{H}_i(x) = -2 \left(\gamma_{ik} \frac{\delta \mathcal{S}}{\delta \gamma_{kj}(x)} \right)_{,j} + \frac{\delta \mathcal{S}}{\delta \gamma_{kl}(x)} \gamma_{kl,i} + \sum_{a=1}^{2} \frac{\delta \mathcal{S}}{\delta \phi_a(x)} \phi_{a,i} , \tag{2.1b}$$

govern the evolution of the generating functional, $\mathcal{S} \equiv \mathcal{S}[\phi_a(x), \gamma_{ij}(x)]$, in *superspace*. For each field configuration $\phi_a(x)$ on a space-like hypersurface with 3-geometry described by the 3-metric $\gamma_{ij}(x)$, the generating functional associates a complex number. Here R denotes the Ricci scalar of the 3-metric, and $V[\phi_a(x)]$ is the scalar field potential. In the ADM formalism, the space-time line element is written as

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = \left(-N^2 + \gamma^{ij}N_iN_j \right) dt^2 + 2N_i dt\, dx^i + \gamma_{ij}dx^i\, dx^j , \tag{2.2}$$

where N and N_i are the lapse and shift functions, respectively. The generating functional is the 'phase' of the wavefunctional in the semiclassical approximation,

$$\Psi \sim e^{i\mathcal{S}} . \tag{2.3a}$$

(The prefactor is neglected here although it has important implications for quantum cosmology [21].) The probability functional

$$\mathcal{P} \equiv |\Psi|^2, \tag{2.3b}$$

is just the square of the wavefunctional (see, e.g., ref. [20]).

The HJ equation (2.1a) and the momentum constraint (2.1b) follow, respectively, from the G^0_0 and G^0_i Einstein equations with the canonical momenta replaced by functional derivatives of $\mathcal{S}$:

$$\pi^{\phi_a}(x) = \frac{\delta \mathcal{S}}{\delta \phi_a(x)} , \quad \text{and} \quad \pi^{ij}(x) = \frac{\delta \mathcal{S}}{\delta \gamma_{ij}(x)} . \tag{2.4}$$

The momentum constraint demands that the generating functional be invariant under an arbitrary change of the spatial coordinates [22] (see also ref. [19], p.1185). The Hamilton-Jacobi equation for general relativity is analogous to the Tomonaga-Schwinger equation that yielded a covariant formulation of quantum electrodynamics [23]. If the generating functional is real, the evolution of the 3-metric for one particular universe is given by

$$\left(\dot{\gamma}_{ij} - N_{i|j} - N_{j|i} \right)/N = 2\gamma^{-1/2} \left(2\gamma_{jk}\gamma_{il} - \gamma_{ij}\gamma_{kl} \right) \frac{\delta \mathcal{S}}{\delta \gamma_{kl}} , \tag{2.5a}$$

whereas the evolution equation for the scalar field is

$$\left(\dot{\phi}_a - N^i \phi_{a,i} \right)/N = \gamma^{-1/2} \frac{\delta \mathcal{S}}{\delta \phi_a} . \tag{2.5b}$$

Here | denotes a covariant derivative with respect to the 3-metric γ_{ij}. Since the lapse and shift function appear neither in the HJ equation (2.1a) nor in the momentum constraint (2.1b), these two equations are *covariant* in that they are valid for all choices of the space-time coordinates. All gauge-dependent quantities which are associated with the lapse and shift appear only in the evolution equations (2.5a,b) for the 3-metric and scalar field.

III. SPATIAL GRADIENT EXPANSION OF THE GENERATING FUNCTIONAL

Superspace describes an ensemble of evolving universes, and its complexity is overwhelming. One may reduce this complexity by expanding the generating functional

$$\mathcal{S} = \sum_{n=0}^{\infty} \mathcal{S}^{(2n)} \tag{3.1}$$

in a series of terms according to the number of spatial gradients that they contain. The great simplification that arises is that one encounters linear partial differential equations at each order for $n \geq 1$. A careful exposition is given of the generating functional of order two. This example illustrates many of the essential aspects required in solving more challenging problems. The consistency of higher order solutions is also demonstrated.

A. Generating Functional of Order Zero

To order zero, the Hamilton-Jacobi equation is

$$\gamma^{-1/2} \left(2\gamma_{ik}\gamma_{jl} - \gamma_{ij}\gamma_{kl} \right) \frac{\delta \mathcal{S}}{\delta \gamma_{ij}}^{(0)} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}}^{(0)} + \sum_{a=1}^{2} \frac{1}{2}\gamma^{-1/2} \left(\frac{\delta \mathcal{S}}{\delta \phi_a}^{(0)} \right)^2 + \gamma^{1/2} V\left(\phi_a \right) = 0. \tag{3.2}$$

In the full HJ eq.(2.1a), the spatial derivative terms appearing in the square brackets have been neglected. They will be recovered at the next order. One attempts a solution of the form,

$$\mathcal{S}^{(0)}[\phi_a(x), \gamma_{ij}(x)] = -2 \int d^3x\, \gamma^{1/2}\, H\left(\phi_a(x) \right). \tag{3.3a}$$

for the zeroth order generating functional. The numerical factor -2 is chosen so that the function of the scalar fields, $H \equiv H(\phi_a)$, corresponds to the usual Hubble parameter in the long-wavelength approximation. Because the integral is over $d^3x\, \gamma^{1/2}$, this functional is invariant under spatial coordinate transformations, and consequently the momentum constraint eq.(2.1b) is satisfied. The HJ eq.(3.2) of order zero holds provided that H satisfies the following nonlinear partial differential equation:

$$H^2 = \frac{2}{3} \sum_{a=1}^{2} \left(\frac{\partial H}{\partial \phi_a} \right)^2 + \frac{1}{3} V\left(\phi_a \right) \ . \tag{3.3b}$$

Since the metric does not appear in this equation, it is often referred to as the separated Hamilton-Jacobi equation (SHJE) (of order zero).

B. Generating Functional of Order Two

The second order HJ equation is

$$\sum_{a=1}^{2} \left[-2\frac{\partial H}{\partial \phi_a} \frac{\delta \mathcal{S}^{(2)}}{\delta \phi_a} \right] + 2H\gamma_{ij}\frac{\delta \mathcal{S}^{(2)}}{\delta \gamma_{ij}} = \frac{1}{2}\gamma^{1/2}R - \sum_{a=1}^{2}\frac{1}{2}\gamma^{1/2}\gamma^{ij}\phi_{a,i}\phi_{a,j}\ . \tag{3.4}$$

It is a linear partial differential equation of the inhomogeneous type. It may be solved using a two step method. (1) One firstly applies the method of characteristics to choose an integration parameter or time parameter; the remainder of the variables are then frozen in time. (2) Secondly, following the work of Salopek and Stewart [4], one makes an ansatz for the second order generating functional which contains all terms with two spatial derivatives. If one wishes to be more direct, one may employ a line integral in superspace [2] which elegantly illuminates the nature of cosmic time.

The characteristic equations

$$\frac{1}{N}\frac{\partial \phi_a}{\partial t} = -2\frac{\partial H}{\partial \phi_a}\,, \tag{3.5a}$$

$$\frac{1}{N}\frac{\partial \gamma_{ij}}{\partial t} = 2H\gamma_{ij} \tag{3.5b}$$

are determined by the coefficients of the functional derivatives, $\delta \mathcal{S}/\delta \phi_a$ and $\delta \mathcal{S}/\delta \gamma_{ij}$, appearing in eq.(3.4). Here the time parameter t as well as the lapse N are arbitrary, but for the sake of simplicity one can safely assume that they are local functions of the scalar fields: $t \equiv t(\phi_a)$, $N \equiv N(\phi_a)$. The above equations may be simplified by performing a conformal transformation,

$$\gamma_{ij}(x) = \Omega^2(\phi_a)\, f_{ij}(x)\,, \tag{3.6}$$

where the conformal factor, $\Omega \equiv \Omega(\phi_a)$, is a function of the scalar fields, and the conformal 3-metric, f_{ij}, is independent of time t. The reduced characteristic equations are then:

$$\frac{1}{N}\frac{\partial \phi_a}{\partial t} = -2\frac{\partial H}{\partial \phi_a}\,, \tag{3.7a}$$

$$\frac{1}{N}\frac{\partial \ln \Omega}{\partial t} = H\,. \tag{3.7b}$$

There are numerous ways to solve these equations. If one is fortunate to obtain a solution for the Hubble function $H \equiv H(\phi_a, \widetilde{\phi}_a)$ which depends on two homogeneous and time-independent parameters $\widetilde{\phi}_a$, $a = 1, 2$, then one may integrate these equations immediately:

$$\Omega(\phi_a) = \left(\frac{\partial H}{\partial \widetilde{\phi}_1}\right)^{-1/3}\,, \tag{3.8a}$$

$$e(\phi_a) = \frac{\partial H}{\partial \widetilde{\phi}_1}\bigg/\frac{\partial H}{\partial \widetilde{\phi}_2}\,, \tag{3.8b}$$

$$f_{ij} = \Omega^{-2}(\phi_a)\, \gamma_{ij}\,. \tag{3.8c}$$

Both $e(\phi_a)$ and f_{ij} are independent of time:

$$\frac{1}{N}\frac{\partial e}{\partial t} = 0\,, \tag{3.9a}$$

$$\frac{1}{N}\frac{\partial f_{ij}}{\partial t} = 0\,. \tag{3.9b}$$

By differentiating the SHJE (3.3b) with respect to the parameters, one may verify that these fields satisfy the reduced characteristic equations (3.7b). Note that this solution refers neither to the time parameter t nor to the lapse N— they are redundant variables.

The integration of the characteristic equations may be viewed as a transformation of the fields

$$(\phi_1, \phi_2, \gamma_{ij}) \to (\Omega, e, f_{ij}). \tag{3.10}$$

Utilizing the conformal 3-metric f_{ij} instead the original 3-metric γ_{ij} is analogous to using comoving coordinates rather than physical coordinates in cosmological systems. $\Omega(x) \equiv \Omega[\phi_a(x)]$ can be interpreted as the integration parameter or 'time' parameter, whereas $e(x), f_{ij}(x)$ are independent of time. The integration parameter could have been chosen to be any function of the scalar fields, but the choice of $t = \Omega \equiv \Omega(\phi_a)$ will simplify the subsequent integration. From the second characteristic equation (3.7b), $\partial \Omega/\partial t = 1$, implies that the lapse is given by $N = 1/(\Omega H)$, and hence the scalar field evolves in Ω according to

$$\left(\frac{\partial \phi_a}{\partial \Omega}\right)_e = -\frac{2}{\Omega H}\frac{\partial H}{\partial \phi_a}\,. \tag{3.11}$$

In some instances, it may be difficult to determine a solution for the Hubble function which depends on two parameters. Alternatively, one may integrate eq.(3.11) where e now refers to an arbitrary and inhomogeneous constant of integration.

Using the new variables, the HJ equation of order two may be rewritten as

$$\Omega H \left. \frac{\delta \mathcal{S}^{(2)}}{\delta \Omega} \right|_{e,f_{ij}} = \frac{1}{2}\gamma^{1/2} R - \sum_{a=1}^{2} \frac{1}{2}\gamma^{1/2}\gamma^{ij}\phi_{a,i}\phi_{a,j} \,. \tag{3.12}$$

The method of characteristics has simplified the left hand side considerably. It now remains to express the right hand side in terms of the new variables. For example, a conformal transformation of the Ricci curvature leads to

$$R(\Omega^2 f_{ij}) = \Omega^{-2}\widetilde{R} - 8\Omega^{-5/2}(\Omega^{1/2})^{;i}_{;i} \tag{3.13}$$

where $\widetilde{R}$ is the Ricci scalar of the conformal 3-metric f_{ij}, and a semi-colon ; denotes a covariant derivative with respect to the conformal 3-metric. As a result, one finds the following final form for the HJ equation of order two,

$$\left. \frac{\delta \mathcal{S}^{(2)}}{\delta \Omega} \right|_{e,f_{ij}} = f^{1/2} \left[\frac{\widetilde{R}}{2H} - \frac{2}{H\Omega}\Omega^{;i}_{;i} + l(\Omega,e)\,\Omega_{;i}\Omega^{;i} + 2m(\Omega,e)\,\Omega_{;i}e^{;i} + n(\Omega,e)e_{;i}e^{;i} \right] \tag{3.14a}$$

where l, m, n are functions of (Ω, e):

$$l(\Omega,e) = \frac{1}{H\Omega^2} - \frac{1}{2H}\left[\left(\frac{\partial\phi_1}{\partial\Omega}\right)_e^2 + \left(\frac{\partial\phi_2}{\partial\Omega}\right)_e^2 \right] , \tag{3.14b}$$

$$m(\Omega,e) = -\frac{1}{2H}\left[\left(\frac{\partial\phi_1}{\partial\Omega}\right)_e \left(\frac{\partial\phi_1}{\partial e}\right)_\Omega + \left(\frac{\partial\phi_2}{\partial\Omega}\right)_e \left(\frac{\partial\phi_2}{\partial e}\right)_\Omega \right] , \tag{3.14c}$$

$$n(\Omega,e) = -\frac{1}{2H}\left[\left(\frac{\partial\phi_1}{\partial e}\right)_\Omega^2 + \left(\frac{\partial\phi_2}{\partial e}\right)_\Omega^2 \right] . \tag{3.14d}$$

1. Ansatz for Second Order Solution

An Ansatz,

$$\mathcal{S}^{(2)} = \int d^3x\, f^{1/2} \left[j(\Omega,e)\widetilde{R} + k_{11}(\Omega,e)\Omega_{;i}\Omega^{;i} + 2k_{12}(\Omega,e)\Omega_{;i}e^{;i} + k_{22}(\Omega,e)e_{;i}e^{;i} \right] . \tag{3.15a}$$

will be used to integrate the HJ equation of order two. j, k_{11}, k_{12} and k_{22}, are functions of (Ω, e) which are determined to be:

$$j(\Omega,e) = \int_0^\Omega d\Omega' \frac{1}{2H(\Omega',e)} + j_0(e) , \tag{3.15b}$$

$$k_{11}(\Omega,e) = \frac{1}{H\Omega} , \tag{3.15c}$$

$$k_{22}(\Omega,e) = \int_0^\Omega d\Omega'\, n(\Omega',e) + k_0(e) , \tag{3.15d}$$

$$k_{12}(\Omega,e) = 0 , \tag{3.15e}$$

where $n(\Omega, e)$ was defined in eq.(3.14d), and $j_0 \equiv j_0(e)$ and $k_0 \equiv k_0(e)$ are arbitrary functions of e.

In order to construct the above solution, one computes the functional derivative with respect to $\Omega(x)$ of the Ansatz eq.(3.15a):

$$f^{-1/2} \left. \frac{\delta \mathcal{S}^{(2)}}{\delta \Omega} \right|_{e,f_{ij}} = \frac{\partial j}{\partial\Omega}\widetilde{R} - \frac{\partial k_{11}}{\partial\Omega}\Omega_{;i}\Omega^{;i} - 2\frac{\partial k_{11}}{\partial e}\Omega_{;i}e^{;i} + \left(\frac{\partial k_{22}}{\partial\Omega} - 2\frac{\partial k_{12}}{\partial e}\right)e_{;i}e^{;i} - 2k_{11}\Omega^{;i}_{;i} - 2k_{12}e^{;i}_{;i} , \tag{3.16}$$

and matches terms with HJ equation (3.14a) of order two. Since no $e^{;i}_{;i}$ term appears in eq.(3.14a), one finds immediately that the cross term k_{12} vanishes. This fortunate circumstance arose because we chose Ω to be our integration parameter. The remaining equations

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$$\left(\frac{\partial j}{\partial \Omega}\right)_e = \frac{1}{2H}, \tag{3.17a}$$

$$k_{11} = \frac{1}{H\Omega}, \tag{3.17b}$$

$$\left(\frac{\partial k_{11}}{\partial \Omega}\right)_e = -l, \tag{3.17c}$$

$$\left(\frac{\partial k_{11}}{\partial e}\right)_\Omega = -m, \tag{3.17d}$$

$$\left(\frac{\partial k_{22}}{\partial \Omega}\right)_e = n, \tag{3.17e}$$

lead to the solution given above. Initially, it appears that this system of equations may be overdetermined since k_{11} appears in the three equations (3.17b-3.17d). It is a minor miracle that given the first equation, the latter two are automatically satisfied.

The integration of the HJ equation of order two is a nontrivial result. The ansatz (3.15a) and the prescription for determining its free functions represent a complete and explicit integration of the problem. Previously, Salopek and Stewart [4] had solved the single field system. The new ingredients required for resolving the two scalar field problem were: (1) the method of characteristics, and (2) choosing $\Omega(x) \equiv \Omega[\phi_a(x)]$ as the integration parameter. Even then, the final form, eqs.(3.15a-e), of the generating functional of order two is valid for *all* choices of the time hypersurface.

2. Line Integrals and Potential Theory

The HJ eq.(3.14a) of order two has the form of an infinite dimensional gradient. It may be integrated directly using a line integral in superspace. The required aspects of potential theory are first briefly reviewed.

The fundamental problem in potential theory is: given a force field $g^i(y_k)$ which is a function of n variables y_k, what is the potential $\Phi \equiv \Phi(y_k)$ (if it exists) whose gradient returns the force field:

$$\frac{\partial \Phi}{\partial y_i} = g^i(y_k) \quad ? \tag{3.18}$$

Not all force fields are derivable from a potential. Provided that the force field satisfies the integrability relation,

$$0 = \frac{\partial g^i}{\partial y_j} - \frac{\partial g^j}{\partial y_i} = \left[\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_i}\right] \Phi, \tag{3.19}$$

(i.e., it is curl-free), one may find a solution which is conveniently expressed using a line integral

$$\Phi(y_k) = \int_C \sum_j d\overline{y}_j \, g^j(\overline{y}_l). \tag{3.20}$$

If the two endpoints are fixed, all contours return the same answer. In practice, we will employ the simplest contour that one can imagine: a line connecting the origin to the observation point y_k. Using s, $0 \le s \le 1$, to parameterize the contour, $\overline{y}_l = sy_l$, the line integral may be rewritten as

$$\Phi(y_k) = \sum_{j=1}^{n} \int_0^1 ds \, y_j \, g^j(\overline{y}_l) + \Phi_0, \tag{3.21}$$

where Φ_0 is independent of y_k.

3. Line Integral Method for Second Order Generating Functional

By identifying, $j \to x$, $\sum_j \to \int d^3x$, $y_j \to \Omega(x)$, $dy_j = ds\Omega(x)$, one may integrate the infinite dimensional gradient eq.(3.14a),

$$\mathcal{S}^{(2)}[\Omega(x), e(x), f_{ij}(x)] = \int d^3x \int_0^1 ds \, \Omega(x) \frac{\delta \mathcal{S}^{(2)}}{\delta \overline{\Omega}} \bigg|_{e, f_{ij}} + \mathcal{S}_0^{(2)}[e(x), f_{ij}(x)], \qquad (3.22)$$

with $\overline{\Omega}(x) = s\Omega(x)$. One recovers the result (3.15a) guessed earlier provided the 'constant functional', $\mathcal{S}_0^{(2)}$, is given by

$$\mathcal{S}_0^{(2)}[e(x), f_{ij}(x)] = \int d^3x f^{1/2} \left[j_0(e)\widetilde{R} + k_0(e)f^{ij}e_{,i}e_{,j} \right], \qquad (3.23)$$

which is independent of $\Omega(x)$. Because of an integrability condition discussed in the next section, this result does not depend on the contour of integration in superspace.

4. Integrability Condition for Generating Functional

A line integral proves useful for computing higher order terms in the spatial gradient expansion.
For $n \geq 1$, one finds that $S^{(2n)}$ satisfies the following inhomogeneous, linear partial differential equation:

$$\widehat{O}(x) \, \mathcal{S}^{(2n)} = -\mathcal{R}^{(2n)}(x). \qquad (3.24a)$$

The differential operator $\widehat{O}(x)$ is given by

$$\widehat{O}(x) \equiv \sum_{a=1}^{2} -2 \frac{\partial H}{\partial \phi_a} \frac{\delta}{\delta \phi_a(x)} + 2H\gamma_{ij} \frac{\delta}{\delta \gamma_{ij}(x)}. \qquad (3.24b)$$

The remainder term $\mathcal{R}^{(2n)}(x)$,

$$\mathcal{R}^{(2n)}(x) = \gamma^{-1/2} \sum_{p=1}^{n-1} \frac{\delta \mathcal{S}^{(2p)}}{\delta \gamma_{ij}(x)} \frac{\delta \mathcal{S}^{(2n-2p)}}{\delta \gamma_{kl}(x)} \left(2\gamma_{jk}\gamma_{li} - \gamma_{ij}\gamma_{kl} \right)$$

$$+ \gamma^{-1/2} \sum_{p=1}^{n-1} \sum_{a=1}^{2} \frac{1}{2} \frac{\delta \mathcal{S}^{(2p)}}{\delta \phi_a(x)} \frac{\delta \mathcal{S}^{(2n-2p)}}{\delta \phi_a(x)} + \mathcal{V}^{(2n)}. \qquad (3.24c)$$

is independent of $\mathcal{S}^{(2n)}$ but it contains contributions from all previous orders — it is assumed to be known. The superspace potential $\mathcal{V}^{(2n)}$ is defined to be

$$\mathcal{V}^{(2n)}(x) = -\frac{1}{2}\gamma^{1/2}R + \sum_{a=1}^{2} \frac{1}{2}\gamma^{1/2}\gamma^{ij}\phi_{a,i}\phi_{a,j} \quad \text{for n} = 1,$$

$$= 0 \quad \text{otherwise.} \qquad (3.24d)$$

By manipulating eq.(3.24a-d), one may compute the following commutator,

$$\left[\widehat{O}(x), \widehat{O}(y) \right] \mathcal{S}^{(2n)} \equiv \widehat{O}(y) \, \mathcal{R}^{(2n)}(x) - \widehat{O}(x) \, \mathcal{R}^{(2n)}(y)$$

$$= [\gamma^{ij}(x) \, \mathcal{H}_j^{(2n-2)}(x) + \gamma^{ij}(x') \, \mathcal{H}_j^{(2n-2)}(x')] \frac{\partial}{\partial x^i} \delta^3(x - x'), \qquad (3.25)$$

which assumes by induction that $\mathcal{S}^{(2)}, \mathcal{S}^{(4)}, \ldots, \mathcal{S}^{(2n-2)}$ satisfy eq.(3.24a). The 'integrability condition' of potential theory, eq.(3.19), demands that the commutator (3.25) vanish. In the above expression, $\mathcal{H}_j^{(2n-2)}$ is the momentum constraint evaluated using the generating functional of order $(2n-2)$:

$$\mathcal{H}_j^{(2n-2)}(x) \equiv -2 \left(\gamma_{jk} \frac{\delta \mathcal{S}^{(2n-2)}}{\delta \gamma_{kl}(x)} \right)_{,l} + \frac{\delta \mathcal{S}^{(2n-2)}}{\delta \gamma_{kl}(x)} \gamma_{kl,j} + \sum_{a=1}^{2} \frac{\delta \mathcal{S}^{(2n-2)}}{\delta \phi_a(x)} \phi_{a,i}. \qquad (3.26)$$

We conclude that $\mathcal{S}^{(2n)}$ is indeed integrable provided the term of previous order, $\mathcal{S}^{(2n-2)}$, is invariant under reparametrizations of the spatial coordinates: $\mathcal{H}_j^{(2n-2)} = 0$. In general, the integrability condition for the Hamilton-Jacobi equation follows from the Poisson brackets [24] between the energy densities evaluated at the two spatial points x and x':

$$\{\mathcal{H}(x), \mathcal{H}(x')\} = [\gamma^{ij}(x)\mathcal{H}_j(x) + \gamma^{ij}(x')\mathcal{H}_j(x')] \frac{\partial}{\partial x^i} \delta^3(x - x') . \tag{3.27}$$

In practice, one would solve the HJ eq.(3.24a) of order 2n by changing variables according to eq.(3.8a-c), and then applying a line integral analogous to eq.(3.22):

$$\mathcal{S}^{(2n)}[\Omega(x), e(x), f_{ij}(x)] = -\int d^3x \int_0^1 \frac{ds}{s} \frac{1}{H\left[\overline{\Omega}(x), e(x)\right]} \mathcal{R}^{(2n)}[\overline{\Omega}(x), e(x), f_{ij}(x)] \tag{3.28}$$

$$+ \mathcal{S}_0^{(2n)}[e(x), f_{ij}(x)], \qquad \text{with } \overline{\Omega}(x) = s\,\Omega(x) . \tag{3.29}$$

Once again, the 'constant functional,' $\mathcal{S}_0^{(2n)}[e(x), f_{ij}(x)]$ depends only on the fields $e(x)$, $f_{ij}(x)$, which are independent of $\Omega(x)$. The constant functional contains all such terms with $2n$ spatial derivatives which are invariant under reparametrizations of the spatial coordinates.

C. The Nature of Cosmic Time

For the semiclassical limit, a line integral in superspace goes a long way towards illuminating the nature of cosmic time. Different contours of integration analogous to eq.(3.29) correspond to different intermediate choices for the integration parameter or time parameter. However, they all yield the same answer for the generating functional provided that spatial gauge-invariance is maintained at each order of the computation.

IV. QUADRATIC CONSTANT APPROXIMATION FOR GRAVITY WITH MULTIPLE SCALAR FIELDS

In the last section, a prescription was given for computing the generating functional to arbitrary order in the spatial gradient expansion. For a single scalar field or a single dust field, this has been done explicitly to 4th order or 6th order, respectively, in spatial gradients [2]. The case for a single dust field is of considerable practical importance because it leads to the Zel'dovich approximation for general relativity [14], [15].

However, for some cosmological applications, a finite number of terms is insufficient. In this section, it will be shown how to effectively sum an infinite subset of terms in the spatial gradient expansion. The 'quadratic constant approximation' generalizes the 'quadratic curvature approximation' developed in an earlier paper [1].

A. Factoring Out the Long-Wavelength Background

Beginning with the full HJ eq.(2.1a), one first subtracts the long-wavelength background from the generating functional:

$$\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{F} , \quad \mathcal{S}^{(0)} = -2\int d^3x \gamma^{1/2} H(\phi_a) , \tag{4.1}$$

where H satisfies the separated Hamilton-Jacobi equation (3.3b). The functional for fluctuations, $\mathcal{F}$, now satisfies

$$\sum_{a=1}^{2} -2\frac{\partial H}{\partial \phi_a} \frac{\delta \mathcal{F}}{\delta \phi_a} + 2H\gamma_{ij} \frac{\delta \mathcal{F}}{\delta \gamma_{ij}} + \gamma^{-1/2}\left[2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x)\right] \frac{\delta \mathcal{F}}{\delta \gamma_{ij}(x)} \frac{\delta \mathcal{F}}{\delta \gamma_{kl}(x)}$$

$$+ \frac{1}{2}\gamma^{-1/2}\sum_{a=1}^{2}\left(\frac{\delta \mathcal{F}}{\delta \phi_a(x)}\right)^2 - \frac{1}{2}\gamma^{1/2}R + \sum_{a=1}^{2}\frac{1}{2}\gamma^{1/2}\gamma^{ij}\phi_{a,i}\phi_{a,j} = 0 . \tag{4.2}$$

The first line of eq.(4.2) may simplified if one introduces the change of variables, $(\phi_a, \gamma_{ij}) \rightarrow (\Omega, e, f_{ij})$, described by eqs.(3.8a-c). Functional derivatives with respect to the fields transform according to

$$\frac{\delta}{\delta\gamma_{ij}} = \Omega^{-2}\left.\frac{\delta}{\delta f_{ij}}\right|_{\Omega,e} \quad, \qquad \frac{\delta}{\delta\phi_a} = \frac{\partial\Omega}{\partial\phi_a}\left(\left.\frac{\delta}{\delta\Omega}\right|_{e,f_{ij}} - \frac{2}{\Omega}f_{ij}\left.\frac{\delta}{\delta f_{ij}}\right|_{\Omega,e}\right) + \frac{\partial e}{\partial\phi_a}\left.\frac{\delta}{\delta e}\right|_{\Omega,f_{ij}} . \tag{4.3}$$

(Henceforth, one suppresses the symbols $|_\Omega$, $|_e$, $|_{f_{ij}}$ which denote the variables that are held constant during functional differentiation.)

The HJ equation reduces to

$$\frac{\delta\mathcal{F}}{\delta\Omega} + \frac{f^{-1/2}}{\Omega^4 H}\left[2f_{il}f_{jk} - f_{ij}f_{kl}\right]\frac{\delta\mathcal{F}}{\delta f_{ij}}\frac{\delta\mathcal{F}}{\delta f_{kl}} + \sum_{a=1}^{2}\frac{f^{-1/2}}{2\Omega^4 H}\left[\frac{\partial\Omega}{\partial\phi_a}\left(\frac{\delta\mathcal{F}}{\delta\Omega} - \frac{2}{\Omega}f_{ij}\frac{\delta\mathcal{F}}{\delta f_{ij}}\right) + \frac{\partial e}{\partial\phi_a}\frac{\delta\mathcal{F}}{\delta e}\right]^2 = \frac{\delta\mathcal{S}^{(2)}}{\delta\Omega} \tag{4.4}$$

where $\mathcal{S}^{(2)}$ was given in eq.(3.15a). The momentum constraint maintains the same form as before but it is now expressed in terms of the new variables (Ω, e, f_{ij}):

$$\mathcal{H}_i(x) = -2\left(f_{ik}\frac{\delta\mathcal{F}}{\delta f_{kj}}\right)_{,j} + \frac{\delta\mathcal{F}}{\delta f_{kl}}f_{kl,i} + \frac{\delta\mathcal{F}}{\delta e}e_{,i} + \frac{\delta\mathcal{F}}{\delta\Omega}\Omega_{,i} = 0 . \tag{4.5}$$

B. Integral form of HJ equation

Following ref. [2], an integral form of the HJ eq.(4.4) may be constructed using a line integral in superspace,

$$\mathcal{F}[\Omega(x), e(x), f_{ij}(x)] + \int d^3x\int_0^1\frac{ds}{s}\frac{f^{-1/2}}{\overline{\Omega}^3 H}\left[2f_{il}f_{jk} - f_{ij}f_{kl}\right]\frac{\delta\mathcal{F}}{\delta f_{ij}}\frac{\delta\mathcal{F}}{\delta f_{kl}} +$$

$$\int d^3x\int_0^1\frac{ds}{s}\sum_{a=1}^{2}\frac{f^{-1/2}}{\overline{\Omega}^3 H}\left[\frac{\partial\Omega}{\partial\phi_a}\left(\frac{\delta\mathcal{F}}{\delta\overline{\Omega}} - 2\overline{\Omega}^{-1}f_{ij}\frac{\delta\mathcal{F}}{\delta f_{ij}}\right) + \frac{\partial e}{\partial\phi_a}\frac{\delta\mathcal{F}}{\delta e}\right]^2 = \mathcal{S}^{(2)}[\Omega(x), e(x), f_{ij}(x)], \tag{4.6}$$

where once again $\overline{\Omega}(x) = s\Omega(x)$.

C. Ansatz

One makes an Ansatz for the fluctuation functional (4.1) of the form,

$$\mathcal{F} = \mathcal{S}^{(2)} + \mathcal{Q}, \tag{4.7}$$

where the *quadratic functional*,

$$\mathcal{Q} = \int d^3x f^{1/2}\left[\widetilde{R}\,\widehat{S}_{11}\,\widetilde{R} + 2\widetilde{R}\,\widehat{S}_{12}\,(\widetilde{D}^2 e) + (\widetilde{D}^2 e)\,\widehat{S}_{22}\,(\widetilde{D}^2 e) + \widetilde{R}^{ij}\,\widehat{T}\,\widetilde{R}_{ij} - \frac{3}{8}\widetilde{R}\,\widehat{T}\,\widetilde{R}\right], \tag{4.8}$$

consists of all quadratic combinations of the fields, $\widetilde{D}^2 e$, $\widetilde{R}$, $\widetilde{R}_{ij}$ (but *not* $\widetilde{D}^2\Omega$), which maintain spatial gauge invariance. $\widetilde{D}^2$ is the Laplacian operator with respect to the conformal 3-metric, e.g.,

$$\widetilde{D}^2\widetilde{R} \equiv \widetilde{D}^{:i}\widetilde{D}_{:i}\widetilde{R} \equiv \widetilde{R}^{:i}{}_{;i} = f^{-1/2}\left(f^{1/2}f^{ij}\widetilde{R}_{,j}\right)_{,i}, \tag{4.9}$$

One interprets the operator $\widehat{T}$ for tensor perturbations to be a Taylor series of the form

$$\widehat{T}(\Omega, e, \widetilde{D}^2) = \sum_n T_n(\Omega, e)(\widetilde{D}^2)^n, \tag{4.10}$$

and similarly for the scalar operator $\widehat{S}_{ab}$, which is a two-by-two symmetric matrix. (The full Riemann tensor $\widetilde{R}_{ijkl}$ does not appear in the Ansatz because for three spatial dimensions because it may be written in terms of the Ricci tensor [25].) Although the first and fifth terms in eq.(4.8) may be combined into a single one, the present form simplifies the

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final evolution equations for $\widehat{S}$ and $\widehat{T}$ (see eqs.(4.16b), (4.17b)). The Ansatz (4.8) generalizes the quadratic curvature approximation considered by Salopek and Stewart [1].

The perturbation rules are the same as those developed in ref. [1]. By first order, we refer to terms such as $\widetilde{R}$, $\widetilde{D}^2\Omega$, $\widetilde{D}^2e$, or $\widetilde{D}^2\widetilde{R}$, $\widetilde{D}^4\Omega$, $\widetilde{D}^4e$, which vanish if the fields are homogeneous; they may contain any number of spatial derivatives. Quadratic terms are a product of two linear terms.

In computing the various functional derivatives, it is useful to note that for a small variation of the conformal 3-metric δf_{ij} the corresponding change in the Ricci tensor is

$$\delta\widetilde{R}_{ij} = \frac{1}{2}f^{kl}\left[\delta f_{li;jk} + \delta f_{lj;ik} - \delta f_{ij;lk} - \delta f_{lk;ij}\right].\tag{4.11}$$

In the integral form of the HJ equation, integration by parts is permitted which simples the analysis considerably. (However, in its differential form (2.1a) one cannot simply discard total spatial derivatives.) In addition, all cubic terms are neglected.

To linear order, the functional derivatives of $\mathcal{S}^{(2)}$, eq.(3.15a), are

$$f^{-1/2}\frac{\delta\mathcal{S}^{(2)}}{\delta\Omega} = \frac{\partial j}{\partial\Omega}\widetilde{R} - 2k_{11}\widetilde{D}^2\Omega\tag{4.12a}$$

$$f^{-1/2}\frac{\delta\mathcal{S}^{(2)}}{\delta f_{ij}} = j\left(\frac{\widetilde{R}}{2}f^{ij} - \widetilde{R}^{ij}\right) + j^{;ij} - j_{;k}^{;k}f^{ij},\tag{4.12b}$$

$$f^{-1/2}\frac{\delta\mathcal{S}^{(2)}}{\delta e} = \frac{\partial j}{\partial e}\widetilde{R} - 2k_{22}\widetilde{D}^2e.\tag{4.12c}$$

and those of the quadratic functional, eq.(4.8), are

$$f^{-1/2}\frac{\delta\mathcal{Q}}{\delta\Omega} = 0,\tag{4.13a}$$

$$f^{-1/2}\frac{\delta\mathcal{Q}}{\delta f_{ij}} = 2\widehat{S}_{11}\left[\widetilde{R}^{;ij} - \widetilde{D}^2\widetilde{R}f^{ij}\right] + 2\widehat{S}_{12}\left[\left(\widetilde{D}^2e\right)^{;ij} - \widetilde{D}^4ef^{ij}\right] +\tag{4.13b}$$

$$\widehat{T}\left[\frac{1}{4}\widetilde{R}^{;ij} + \frac{1}{4}\widetilde{D}^2\widetilde{R}f^{ij} - \widetilde{D}^2\widetilde{R}^{ij}\right],\tag{4.13c}$$

$$f^{-1/2}\frac{\delta\mathcal{Q}}{\delta e} = 2\widetilde{D}^2\left[\widehat{S}_{12}\widetilde{R} + \widehat{S}_{22}\widetilde{D}^2e\right].\tag{4.13d}$$

At the present level of approximation, $\widetilde{D}^2$ commutes with any function of (Ω, E), e.g., $\widetilde{D}^2S_{11} = S_{11}\widetilde{D}^2$, etc. After a straightforward computation (which is very similar to that of ref. [1]), one obtains the tensor equation

$$\frac{\partial\widehat{T}}{\partial\ln\Omega} + \frac{2}{\Omega^3 H}\left(j + \widehat{T}\widetilde{D}^2\right)^2 = 0,\quad\text{(tensor)}\tag{4.14}$$

and the scalar equation

$$\frac{\partial\widehat{S}}{\partial\ln\Omega} + 2\widetilde{D}^4\,\widehat{S}\widehat{A}\widehat{S} + \widetilde{D}^2\,\widehat{C}^T\widehat{S} + \widetilde{D}^2\,\widehat{S}\widehat{C} + \frac{1}{2}\widehat{B} = 0,\quad\text{(scalar)}\tag{4.15a}$$

where $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ are operator matrices. (From now on, the operator symbol $\widehat{}$ will be suppressed.) They are given by

$$A = \frac{1}{\Omega^3 H}\left[\vec{a}_1\vec{a}_1^T + \vec{a}_2\vec{a}_2^T\right],\tag{4.15b}$$

$$B = \frac{1}{\Omega^3 H}\left[\vec{b}_1\vec{b}_1^T + \vec{b}_2\vec{b}_2^T\right],\tag{4.15c}$$

$$C = \frac{1}{\Omega^3 H}\left[\vec{a}_1\vec{b}_1^T + \vec{a}_2\vec{b}_2^T\right].\tag{4.15d}$$

Here T denotes the transpose of a matrix or a vector, and the vectors $\vec{a}_1$, $\vec{a}_2$ and $\vec{b}_1$, $\vec{b}_2$ are

$$\vec{a}_1^T = \left[4\frac{\partial \ln \Omega}{\partial \phi_1}, \ \frac{\partial e}{\partial \phi_1}\right], \tag{4.15e}$$

$$\vec{a}_2^T = \left[4\frac{\partial \ln \Omega}{\partial \phi_2}, \ \frac{\partial e}{\partial \phi_2}\right], \tag{4.15f}$$

$$\vec{b}_1^T = \left[\left(\frac{\partial \Omega}{\partial \phi_1}\right)\left(\frac{\partial j}{\partial \Omega} - \frac{j}{\Omega}\right) + \frac{\partial j}{\partial e}\frac{\partial e}{\partial \phi_1}, \ 4\frac{\partial \ln \Omega}{\partial \phi_1}\frac{\partial j}{\partial e} - 2k_{22}\frac{\partial e}{\partial \phi_1}\right], \tag{4.15g}$$

$$\vec{b}_2^T = \left[\left(\frac{\partial \Omega}{\partial \phi_2}\right)\left(\frac{\partial j}{\partial \Omega} - \frac{j}{\Omega}\right) + \frac{\partial j}{\partial e}\frac{\partial e}{\partial \phi_2}, \ 4\frac{\partial \ln \Omega}{\partial \phi_2}\frac{\partial j}{\partial e} - 2k_{22}\frac{\partial e}{\partial \phi_2}\right], \tag{4.15h}$$

The tensor equation is a single nonlinear ordinary differential equation of the Riccati type. The scalar equation is also of the Riccati type, but the dependent variable S_{ab} is a 2-by-2 symmetric matrix. An additional complications arise in the latter case because the coefficients A, B, C, appearing in eq.(4.15a) need not be commuting.

D. Reduction of Riccati Equations

The Riccati equations, (4.14) and (4.15a), may be reduced to linear ordinary differential equations. For the tensor perturbation, one defines the Riccati transformation, $T(\Omega, e, \widetilde{D}^2) \to y(\Omega, e, \widetilde{D}^2)$,

$$\widehat{T} = \frac{H\Omega^3}{2\widetilde{D}^4}\frac{1}{y}\frac{\partial y}{\partial \ln \Omega} - \widetilde{D}^{-2}j, \quad \text{(tensor perturbation)} \tag{4.16a}$$

which leads to the linear equation for a massless field :

$$0 = \frac{\partial^2 y}{\partial(\ln \Omega)^2} + \left[3 + \frac{1}{H}\frac{\partial H}{\partial \ln \Omega}\right]\frac{\partial y}{\partial \ln \Omega} - \frac{1}{H^2\Omega^2}\widetilde{D}^2 y. \quad \text{(tensor perturbation)} \tag{4.16b}$$

Because non-commuting matrices appear in the scalar case, one must be careful about the order of various matrices, but otherwise the method is the same as the previous case. By defining the matrix Riccati transformation, $S_{ab}(\Omega, e, \widetilde{D}^2) \to W_{ab}(\Omega, e, \widetilde{D}^2)$,

$$S = \frac{1}{2\widetilde{D}^4}\,A^{-1}\,\frac{\partial W}{\partial \ln \Omega}\,W^{-1} - \frac{1}{2\widetilde{D}^2}A^{-1}C, \quad \text{(scalar perturbation)} \tag{4.17a}$$

one recovers the linear matrix equation for the scalar perturbation,

$$0 = \frac{\partial^2 W}{\partial(\ln \Omega)^2} + A\frac{\partial A^{-1}}{\partial \ln \Omega}\,\frac{\partial W}{\partial \ln \Omega} - \frac{1}{H^2\Omega^2}\widetilde{D}^2 W. \quad \text{(scalar perturbation)} \tag{4.17b}$$

The case of a single scalar field has been much studied. Mukhanov *et al* [26] derived the scalar perturbation equation by considering quadratic perturbations of the Einstein action. Hawking, Laflamme and Lyons [27] utilized this equation in their discussion of the arrow of time. Using the Einstein field equations, Hwang [28] gave an elegant interpretation in terms of uniform curvature gauge. Salopek and Stewart [1] used the scalar equation to discuss microwave anisotropies and galaxy correlations arising from inflation. Deruelle *et al* [29] employed an equation which is basically equivalent to eq.(4.17b) in their discussion of extended inflation [12] which incorporates two scalar fields. A path integral approach for multiple fields was considered in ref. [30]. However, the present form of the scalar perturbation equation (4.17b) possesses some distinct advantages over earlier formulations.

Because there are no mass terms appearing on the right-hand-side, eq.(4.17b) is a particularly elegant form for the scalar perturbation equation. For example, in the large wavelength limit, one may neglect the last term appearing on the right-hand-side of eq.(4.17b). For an inflationary epoch, W then approaches a constant,

$$W(\Omega, e, \tilde{D}^2) \to W^\infty(e, \tilde{D}^2) \tag{4.18}$$

(*i.e.*, it is independent of the integration parameter $\ln \Omega$). The numerical value of W^∞ and how it depends on $\widetilde{D}^2$ are of high importance for observational cosmology. Approximate results will be given in the next section.

V. INITIAL CONDITIONS

It is difficult to obtain analytic solutions of the scalar perturbation equation (4.17b) for two scalar fields. Numerical computations have been considered in an earlier work [10]. An approximate analytic method which is well known for the case of a single scalar field case will be extended to include multiple fields. Long wavelength and short wavelength perturbations will be matched at the Hubble radius. (Solutions of the tensor evolution equation (4.16b) and their applications to cosmology have already been examined, [1], [9], and they will not be discussed further in this section.)

During a period of cosmological inflation, the initial conditions for the tensor perturbation y and the scalar perturbation matrix W are set by quantum conditions: for wavelengths that are much shorter than the Hubble radius, $1/H$, one assumes that each mode began in the ground state.

It is useful to rewrite the scalar evolution equation as a set of first order differential equations. One first introduces a momentum variable P, which is a matrix defined by

$$P = A^{-1} \frac{\partial W}{\partial \ln \Omega} \,. \tag{5.1}$$

The evolution equation for the pair (W, P) is then

$$\frac{\partial W}{\partial \ln \Omega} = AP \,, \tag{5.2a}$$

$$\frac{\partial P}{\partial \ln \Omega} = \frac{1}{H^2 \Omega^2} \widetilde{D}^2 A^{-1} W \,. \tag{5.2b}$$

In general, (W, P) may be complex.

A. Symmetry Conditions

From the onset, it has been assumed that the scalar matrix S appearing in eq.(4.15a) is symmetric. This requirement will impose several conditions on the fields appearing in the Riccati equation (4.17a). Since

$$\vec{a}_1^T \vec{b}_2 = \vec{b}_1^T \vec{a}_2 \,, \tag{5.3}$$

one may show that $A^{-1} C$ is indeed symmetric. PW^{-1} is symmetric provided that

$$W^T P - P^T W = 0 \,. \tag{5.4}$$

Using the first order evolution equation (5.2b), one verifies that the derivative of the left-hand-side with respect to $\ln \Omega$ indeed vanishes. Hence, one typically imposes eq.(5.4) as an initial condition.

B. Conjugate Conditions

In addition, it is possible to arrange that (W, P) are canonically conjugate. The conjugate conditions are

$$WW^\dagger - W^* W^T = 0 \,, \tag{5.5a}$$

$$PP^\dagger - P^* P^T = 0 \,, \tag{5.5b}$$

$$WP^\dagger - W^* P^T = -iI \,, \tag{5.5c}$$

where $\dagger$ represents the Hermitean conjugate, $W^\dagger = (W^*)^T$, and I is the 2-by-2 identity matrix, $I_{ij} = \delta_{ij}$. If these conditions are met at one time, then they are true for all times. For example, one may thus verify that the derivative of the left-hand-side with respect to $\ln \Omega$ vanishes (weakly).

Quantum considerations imply the conjugate conditions, which legislate that (W, P) are intrinsically complex since $i = \sqrt{-1}$ appears in eq.(5.5c). The conjugate conditions will be used to prepare the cosmological system in the ground state at short wavelengths (see Sec. C).

The probability functional (2.3b) for scalar perturbations is determined to be

$$\mathcal{P} = \exp -\frac{1}{2} \int d^3x f^{1/2} \, [\widetilde{D}^{-2} \widetilde{R}, e] \, (WW^\dagger)^{-1} \, [\widetilde{D}^{-2} \widetilde{R}, e]^T \,. \tag{5.6}$$

This functional is peaked about a flat and homogeneous field configuration: $\widetilde{R} = 0 = \widetilde{D}^2 e$. For small deviations about this configuration, the 2-point correlation functions are found to be:

$$< \left[\widetilde{D}^{-2}\widetilde{R}(x)\right]\left[\widetilde{D}^{-2}\widetilde{R}(y)\right] >= f^{-1/2}(x) \; \left[WW^\dagger(x)\right]_{11} \; \delta^3(x-y) \,, \tag{5.7}$$

$$< \left[\widetilde{D}^{-2}\widetilde{R}(x)\right] e(y) >= f^{-1/2}(x) \; \left[WW^\dagger(x)\right]_{12} \; \delta^3(x-y) \,, \tag{5.8}$$

$$< e(x) \, e(y) >= f^{-1/2}(x) \; \left[WW^\dagger(x)\right]_{22} \; \delta^3(x-y) \,. \tag{5.9}$$

(Eq.(5.6) follows from the Riccati equation (4.17a) and the conjugate conditions (5.5a-c):

$$S - S^* = \frac{1}{2\widetilde{D}^4} A^{-1} \left[\frac{\partial W}{\partial \ln \Omega} W^{-1} - \frac{\partial W^*}{\partial \ln \Omega} \left(W^{-1}\right)\right] \tag{5.10}$$

$$= \frac{1}{2\widetilde{D}^4} \left[PW^\dagger \left(W^\dagger\right)^{-1} W^{-1} - P^* W^T \left(W^T\right)^{-1} \left(W^{-1}\right)^*\right] \,, \quad etc.; \tag{5.11}$$

in addition, $A^{-1}C$ was assumed to be real.)

C. Short-Wavelength Behavior

One of the main assumptions of the inflationary model is that short wavelength fluctuations originated in the ground state. Analytic expressions are given for two scalar fields in such a state.

The Laplacian operator $\widetilde{D}^2$ for the conformal 3-metric f_{ij} appears throughout this work. Much of the subsequent operator analysis is simplified by introducing eigenfunctions $u(x,k)$ with eigenvalues $-k^2$:

$$\widetilde{D}^2 u(x,k) = -k^2 u(x,k) \,. \tag{5.12}$$

In the basis of the eigenvectors, the operator W describing the ground state [10] evolves according to

$$W(\Omega, e, k) = \frac{1}{\sqrt{2k\Omega}} \, e^{ik\tau} \, X^{-1} \,, \quad k/(H\Omega) >> 1 \,, \tag{5.13}$$

for wavelengths much shorter than the Hubble radius, $k/(H\Omega) >> 1$. Conformal time τ is given by

$$\tau = \int^{\Omega} d\overline{\Omega} \, \overline{\Omega}^{-2} \, \frac{1}{H(\overline{\Omega}, e)} \,, \tag{5.14}$$

and the components of the transformation matrix X are

$$X_{11} = \frac{1}{4} \frac{\partial \phi_1}{\partial \ln \Omega} \,, \quad X_{12} = \frac{\partial \phi_1}{\partial e} \,, \tag{5.15a}$$

$$X_{21} = \frac{1}{4} \frac{\partial \phi_2}{\partial \ln \Omega} \,, \quad X_{22} = \frac{\partial \phi_2}{\partial e} \,. \tag{5.15b}$$

X is actually the Jacobi matrix for the change of variables, $(4 \ln \Omega, e) \to (\phi_1, \phi_2)$. The inverse X^{-1}

$$(X^{-1})_{11} = 4\frac{\partial \ln \Omega}{\partial \phi_1} \,, \quad (X^{-1})_{12} = 4\frac{\partial \ln \Omega}{\partial \phi_2} \,, \tag{5.16a}$$

$$(X^{-1})_{21} = \frac{\partial e}{\partial \phi_1} \,, \quad (X^{-1})_{22} = \frac{\partial e}{\partial \phi_2} \,, \tag{5.16b}$$

is the Jacobi matrix for the inverse transformation, $(\phi_1, \phi_2) \to (4 \ln \Omega, e)$. Since

$$A = \frac{1}{\Omega^3 H} \, X^{-1}(X^{-1})^T \tag{5.17}$$

follows from eq.(4.15b), one may readily verify the symmetry condition (5.4) and the conjugate conditions (5.5a-c).

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D. Matching Long and Short Wavelength Fluctuations

The short-wavelength approximation breaks down when the comoving wavelength equals the Hubble radius, $k/(H\Omega) \sim 1$, at which time $W(\Omega, e, k)$ quickly approaches a constant value $W^\infty(e, k) \equiv W(\infty, e, k)$ given approximately by

$$W^\infty(e, k) \sim \frac{H}{\sqrt{2k^3}} \, X^{-1}\Big|_{k=H\Omega} . \tag{5.18}$$

$W^\infty(e, k)$ contains virtually all of the observable properties of the cosmological model.

For a single scalar field, $\phi_1 = \phi$, one obtains the well-known expression for the variable zeta, ζ:

$$\zeta(k) \equiv \frac{3}{4} W^\infty(k) = \frac{3H}{\sqrt{2k^3}} \frac{\partial \ln \Omega}{\partial \phi} \bigg|_{k=H\Omega} = \frac{3H^2}{\sqrt{2k^3}} \Big/ \left(\frac{\dot\phi}{N}\right) \bigg|_{k=H\Omega} . \tag{5.19}$$

The power spectrum for ζ is

$$\mathcal{P}_\zeta(k) \equiv \frac{k^3}{2\pi^2} |\zeta(k)|^2 = \frac{9}{4\pi^2} H^4 \Big/ \left(\frac{\dot\phi}{N}\right)^2 \bigg|_{k=H\Omega} . \tag{5.20}$$

For two scalar fields, the power spectrum for ζ is

$$\mathcal{P}_\zeta(k) \equiv \frac{9}{16} \frac{k^3}{2\pi^2} \left[W^\infty (W^\infty)^\dagger\right]_{11} . \tag{5.21}$$

Examples for two scalar fields will be given in the next section.

For scalar perturbations, the resulting cosmic microwave background temperature anisotropy is related to zeta through

$$\left(\frac{\Delta T(x)}{T}\right)_S = -\zeta(x)/15 , \quad |x| = 11,700 \text{Mpc} \tag{5.22}$$

where $|x| = 11,700$ Mpc is the comoving radius for the surface of last scattering, assuming that the present Hubble parameter is $H_0 = 50$ km / s/ Mpc.

VI. DOUBLE INFLATION POWER SPECTRUM

As an example of the quadratic curvature approximation, one computes fluctuations arising from inflation that give rise to galaxy clustering as well as microwave background anisotropies. Consider the potential for two scalar fields,

$$V(\phi_a) = V_1(\phi_1) + V_2(\phi_2) , \tag{6.1}$$

which is the sum of two separable potentials, $V_1(\phi_1)$ and $V_2(\phi_2)$. This potential has been used to describe the double inflation model, where ϕ_1 dominates the energy density of the first epoch of inflation, and ϕ_2 dominates the second epoch.

Although exact solutions of the Hubble function $H(\phi_a)$ exist [5], for illustrative purposes, I will be content to utilize the well-known slow-roll approximation,

$$H_{SR}(\phi_a) = \left[\frac{V_1(\phi_1) + V_2(\phi_2)}{3}\right]^{1/2} , \tag{6.2}$$

where one neglects partial derivatives of the Hubble function in the SHJE (3.3b). This is one of those cases were finding a solution depending on two homogeneous parameters is very difficult, and I will instead integrate eq.(3.11). One may verify the solution:

$$\ln \Omega = -\int_0^{\phi_1} d\phi_1' \frac{V_1(\phi_1')}{\frac{\partial V_1}{\partial \phi_1'}} - \int_0^{\phi_2} d\phi_2' \frac{V_2(\phi_2')}{\frac{\partial V_2}{\partial \phi_2'}} , \tag{6.3a}$$

$$e = \int^{\phi_1} \frac{d\phi_1'}{\frac{\partial V_1}{\partial \phi_1'}} - \int^{\phi_2} \frac{d\phi_2'}{\frac{\partial V_2}{\partial \phi_2'}} . \tag{6.3b}$$

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The inverse transformation matrix is

$$(X^{-1})_{11} = -4\frac{V_1}{\frac{\partial V_1}{\partial \phi_1}}, \quad (X^{-1})_{12} = -4\frac{V_2}{\frac{\partial V_2}{\partial \phi_2}}, \tag{6.4a}$$

$$(X^{-1})_{21} = \frac{1}{\frac{\partial V_1}{\partial \phi_1}}, \quad (X^{-1})_{22} = -\frac{1}{\frac{\partial V_2}{\partial \phi_2}}. \tag{6.5}$$

In the slow roll approximation, $k = H\Omega$ may be approximated by $k \propto \Omega$. The power spectrum, $\mathcal{P}_\zeta(k)$, eq.(5.21), is then given by:

$$\mathcal{P}_\zeta(k) = \frac{3}{4\pi^2}(V_1 + V_2)\left[\left(V_1\Big/\frac{\partial V_1}{\partial \phi_1}\right)^2 + \left(V_2\Big/\frac{\partial V_2}{\partial \phi_2}\right)^2\right], \tag{6.6}$$

$$\ln\left[k/(10^{-4}\mathrm{Mpc}^{-1})\right] = 60 - \int_0^{\phi_1} d\phi_1' \frac{V_1(\phi_1')}{\frac{\partial V_1}{\partial \phi_1'}} - \int_0^{\phi_2} d\phi_2' \frac{V_2(\phi_2')}{\frac{\partial V_2}{\partial \phi_2'}}, \tag{6.7}$$

$$e = \int^{\phi_1} \frac{d\phi_1'}{\frac{\partial V_1}{\partial \phi_1'}} - \int^{\phi_2} \frac{d\phi_2'}{\frac{\partial V_2}{\partial \phi_2'}}. \tag{6.8}$$

It is assumed that the scale $k_0 = 10^{-4}$Mpc experienced 60 e-foldings of inflation after it crossed the Hubble radius during inflation. For simplicity, e is assumed to be a homogeneous parameter. (Inhomogeneities in e lead to isocurvature perturbations which have been considered by Sasaki and Yokoyama [31].)

These analytic results verify the numerical simulations of ref. [10]. Heuristic arguments for these results have been given previously in refs. [32] and [6]. The double inflation model is strongly constrained by cosmological observations [33].

The case of quadratic scalar field potentials,

$$V_1(\phi_1) = \frac{1}{2}m_1^2\phi_1^2, \quad V_2(\phi_2) = \frac{1}{2}m_2^2\phi_2^2, \tag{6.9}$$

is particularly easy to implement. In Fig.(1), power spectra for zeta are plotted for two models: inflation with a single scalar field, and inflation with two scalar fields. Both models have been normalized using COBE's 2-year data set [34]: $\sigma_{sky}(10^0) \equiv [< (\Delta T)^2 >]^{1/2} = 30.5 \pm 2.7\mu K$. Using the methods of refs. [1], [9], a small correction has been made for gravitational waves. For the single field model, the potential is $V(\phi) = m^2\phi^2/2$, with $m = 5.45 \times 10^{-6}$ (recall that $m_P = \sqrt{8\pi}$). For the double field model, the potential is given by eq.(6.9), with $m_1 = 1.435 \times 10^{-5}$ and $m_2 = m_1/6$.

In Fig.(2), the observed galaxy data are displayed using the power spectrum for the density field, $\delta = \delta\rho/\rho$. The data were compiled using several galaxy surveys [35], and it has been assumed that infrared galaxies trace the mass distribution, i.e., the bias factor $b_{IRAS} = 1$ is unity. For a single field, the theoretical curve is unsatisfactory because it substantially overestimates the power at short scales. For the double inflation model, the fit is satisfactory provided the parameters have been carefully chosen. Peter et al [33] have considered additional cosmological tests. These models are the subject of a continuing investigation.

VII. CONCLUSIONS

In elementary applications to mechanics, Hamilton-Jacobi theory is useful because it enables one to compute the constants of integration of a dynamical system. This useful feature becomes even more attractive when one considers Hamilton-Jacobi theory for general relativity. Einstein's field equations contain two constraints which legislate that the energy density and the momentum density vanish. These constraints are, in fact, constants of integration for the remaining evolution equations. Hence by solving the Hamilton-Jacobi equation for general relativity, one solves for the constraints as well as the other constants of integration which include, for example, zeta, eq.(5.21), for multiple scalar fields.

Another attractive feature of HJ theory for general relativity is that it yields a covariant formulation of gravity. One may perform calculations without selecting an explicit choice for the temporal and spatial coordinates. In fact, the generating functional is manifestly invariant under transformations of the spatial coordinates since it is constructed by integrating the Ricci tensor and spatial derivatives of the various scalar fields over the 3-volume. The invariance

of the generating functional under a change of time hypersurface is more subtle. Using a spatial gradient expansion, one may express the generating functional as a line integral in superspace. Each contour of integration in the line integral corresponds to a specific choice of time hypersurface, but all choices yield the same answer. An integrability condition is met because the generating functional is invariant under spatial coordinate transformations.

For two scalar fields, one employs the method of characteristics to transform the old variables (ϕ_1, ϕ_2, f_{ij}) to new ones (Ω, e, f_{ij}). The integration parameter is arbitrary, but the choice of the conformal factor $\Omega \equiv \Omega(\phi_a)$ simplifies the subsequent integration. The remaining new variables e and f_{ij} are constants in the long-wavelength limit. One may sum an infinite subset of terms in the spatial gradient expansion by making an ansatz for the generating functional which is quadratic in the fields $\widetilde{D}^2 e$, $\widetilde{R}$ and $\widetilde{R}_{ij}$.

HJ methods have proved particularly useful in applications to cosmology. As an illustration of the theory, the fluctuation spectra for double inflation was computed analytically confirming previous work. One may adjust the double inflation model to give an adequate explanation of galaxy clustering as well as large angle microwave background fluctuations.

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VIII. FIGURE CAPTIONS

Fig.(1): Power spectra for zeta are shown for two models: (1) standard single field inflation and (2) double inflation. Both utilize quadratic scalar field potentials. The spectra are normalized using large angle microwave background measurements. A small correction has been made for gravitation radiation. Double inflation gives less power at short scales.

Fig.(2): The power spectrum for the observed density field $\delta = \delta\rho/\rho$ is shown. Standard single field inflation gives a poor fit of the data at shorter distances, $k < 10^{-1.4}$ Mpc^{-1}. By adjusting several free parameters, double inflation removes this problem.

POWER SPECTRA FOR ZETA

$\log_{10}[\ \mathcal{P}_\zeta(k)\ ]$

Single Inflation with
$V(\phi) = \frac{1}{2}m^2\phi^2$

Double Inflation with
$V(\phi_1, \phi_2) = \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}m_2^2\phi_2^2$

-7.6

-7.8

-8

-8.2

-8.4

-4   -3   -2   -1   0   1   2

$\log_{10}[\ k\ (\text{Mpc}^{-1})\ ]$

POWER SPECTRA FOR DENSITY PERTURBATION

Single Inflation with
$V(\phi) = \frac{1}{2} m^2 \phi^2$

$\log_{10}[\ \mathcal{P}_\delta(k)\ ]$

Double Inflation with
$V(\phi_1, \phi_2) = \frac{1}{2} m_1^2 \phi_1^2 + \frac{1}{2} m_2^2 \phi_2^2$

$\log_{10}[\ \mathrm{k}\ (\mathrm{Mpc}^{-1})\ ]$